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# Contributions to the theory of diffraction of electromagnetic waves by spherical particles. II

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With 2 figures in the text

## 1. Introduction

In a previous paper[1] a new method of attacking the numerical evaluation problem of diffraction of electromagnetic waves by spherical particles has been presented by the author. In the following it will be referred to as *A*. Here, the phenomenon in the forward direction will be studied in some detail. In *A* only the term interpreted as describing the diffracted radiation was computed for that direction; the result (eq. (*A* 102)):

$$\left.\begin{aligned} E''_{\vartheta_{01}} &= i\,\varrho^2 \cos\varphi \cdot \frac{e^{-i k_0 r}}{k_0 r} \cdot \frac{J_1(y)}{y}; \\ E''_{\varphi_{01}} &= \operatorname{tg}\varphi \cdot E''_{\vartheta_{01}}; \\ y &= \varrho\,\Theta \end{aligned}\right\}$$

agreed with results won by elementary methods.

The practical consequence of (*A* 102) is a much used method of determining the size of spherical particles, e. g. in mists and clouds, by observations on the minima of intensity in the forward direction.[2] These minima lie in directions determined by the zeros of the Bessel function[3]:

$$J_1(\varrho\,\Theta) = 0; \tag{1}$$

or explicitly, for the first few:

$$y_1 = \varrho\,\Theta_1 = 3.832;\ y_2 = \varrho\,\Theta_2 = 7.016;\ y_3 = \varrho\,\Theta_3 = 10.173. \tag{1 a}$$

For large particles this theory is quite satisfactory, but both MECKE and KÖHLER (loc. cit.) emphasize that for decreasing radius of the sphere it must meet with grave objections. Mecke's theoretical investigation, using geometrical optical methods, does not give an ultimate answer to the question when eq. (1) can be trusted. Especially, Köhler uses (1) far below the vague limit given by Mecke ($\varrho \sim 50$—$70$; our notations), and obviously this causes no difficulties. In this paper we will reexamine the problem, starting from an exact solution of Maxwell's equations.

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[1] LJUNGGREN, Ark. Mat. Astr. Fys., *36* A, n:o 14, 1948.

[2] See e. g. KÖHLER, Untersuchungen über die Elemente des Nebels und der Wolken, Diss., 1925; MECKE, Ann. d. Ph., *61*, p. 471 & *62*, p. 623, 1920.

[3] In elementary theory we have (cf Handbuch der Physik, XX, p. 179) sin $\Theta$ instead of $\Theta$. As is pointed out in *A*, pp. 33 & 34, the argument $\varrho\,\Theta$ is a consequence of a better approximation of the Legendre functions.

As has been mentioned in *A*, this investigation implies a discussion of the reflected and the refracted radiation, that is of the terms $C_{02}$ and $C_1$ for $\Theta \sim 0$; the application of the saddle point method encounters the difficulty with a saddle point in the vicinity of the limits of integration. These two cases are, however, not equivalent; the treatment of the reflected radiation ($C_{02}$) will be more intricate, as we will then receive the essential contribution from the region of the integration interval in which the asymptotic behaviour of the cylinder functions changes.

## 2. The refracted radiation

We start with (*A* 25), applying (*A* 20) to get a description valid for $\Theta = 0$:

$$C^{(2)}_{A1} = 4n \sum_{N=-\infty}^{\infty} \int_{-\frac{1}{2}}^{\varrho-\frac{1}{2}} \frac{e^{-2\pi i \xi N}(\xi+\frac{1}{2})}{\xi(\xi+1)} \cdot [q_{1\xi}(n\varrho) - n\,q_{2\xi}(\varrho)]^{-2} \cdot$$
$$\cdot [\zeta^{(2)}_{\xi}(\varrho)\,\zeta^{(1)}_{\xi}(n\varrho)]^{-2} P^{(1)\prime}_{\xi}(\cos\Theta)\sin\Theta\, d\xi. \qquad (2)$$

The term $C^{(2)}_{B1}$ is obtained through the exchange:

$$[q_{1\xi}(n\varrho) - n\,q_{2\xi}(\varrho)]^{-2} \to -[n\,q_{1\xi}(n\varrho) - q_{2\xi}(\varrho)]^{-2};$$

and the terms with an upper index (1) through another exchange:

$$P^{(1)\prime}_{\xi}(\cos\Theta)\sin\Theta \to -\frac{P^{(1)}_{\xi}(\cos\Theta)}{\sin\Theta};$$

Eq:s (*A* 9) then yield the field components:

$$\left.\begin{aligned} E''_{\vartheta} &= -\frac{i\cos\varphi}{k_0 r} e^{-ik_0 r}(C^{(2)}_{A} - C^{(1)}_{B}) \\ E''_{\varphi} &= -\frac{i\sin\varphi}{k_0 r} e^{-ik_0 r}(C^{(1)}_{A} - C^{(2)}_{B}). \end{aligned}\right\}$$

The Debye approximations of the cylinder functions (cf. eq:s (*A* 26)—(*A* 30)) and recurrence formulae for spherical harmonics (cf. (*A* 32)) now change (2) into:

$$C^{(2)}_{A1} = -\frac{1+\cos^2\Theta}{2\cos\Theta} C^{(1)}_{A1} + 2n\varrho^2 \sum_{N=-\infty}^{\infty} (-1)^N \int_{0}^{\frac{\pi}{2}} \frac{\sin^2\tau_1 \sin\tau_2 \cos\tau_1}{(n\sin\tau_1 + \sin\tau_2)^2} \cdot$$
$$\cdot S\,u_{\xi}(\Theta)\, e^{2i\varrho[\sin\tau_1 - n\sin\tau_2 + \cos\tau_1(\tau_2-\tau_1-\pi N)]}\, d\tau_1; \qquad (2\text{ a})$$

where:

$$\left.\begin{aligned} S u_\xi(\Theta) &= \frac{1}{\cos\,\Theta}\left[P_{\xi-1}(\cos\,\Theta) + P_{\xi+1}(\cos\,\Theta)\right]; \\ \cos\,\tau_1 = n\,\cos\,\tau_2 &= \frac{\xi+\frac{1}{2}}{\varrho}. \end{aligned}\right\} \tag{3}$$

For $S u_\xi(\Theta)$ we use an approximate integral representation, easily deduced from ($A$ 97):

$$S u_\xi(\Theta) \approx \frac{4}{\pi}\left(1+\frac{\Theta^2}{3}\right)\int\limits_0^{\frac{\pi}{2}} \cos\,(\varrho\,\Theta\,\cos\,\tau_1\,\cos\,\psi)\,d\psi; \tag{4}$$

valid for small values of $\Theta$ $\left(\Theta < \frac{\pi}{6}\right)$ and especially appropriate for small values of $\xi$. This is just what we want as the saddle point is situated near $\tau_1 = \frac{\pi}{2}$, i. e. $\xi = 0$. We further put:

$$\frac{1+\cos^2\,\Theta}{2\,\cos\,\Theta} \approx 1;$$

which is also a permissible approximation for $\Theta < \frac{\pi}{6}$; the error is then about 1 % at most. The introduction of these approximations transforms (2 a) to:

$$C_{A1}^{(2)} + C_{A1}^{(1)} = \frac{4\,n\,\varrho^2}{\pi}\left(1+\frac{\Theta^2}{3}\right)\sum_{N=-\infty}^{\infty}(-1)^N \int\limits_0^{\frac{\pi}{2}} d\psi \int\limits_0^{\frac{\pi}{2}} \frac{\sin^2\,\tau_1\,\sin\,\tau_2\,\cos\,\tau_1}{(n\,\sin\,\tau_1 + \sin\,\tau_2)^2}\cdot \left[e^{i\varrho\,G_{+}^{(1)}} + e^{i\varrho\,G_{-}^{(1)}}\right]d\,\tau_1; \tag{5}$$

where:

$$G_{+,-}^{(1)} = \cos\,\tau_1\left[-2\,\pi N - 2\,\tau_1 + 2\,\tau_2 \pm \Theta\,\cos\,\psi\right] + 2\,\sin\,\tau_1 - 2\,n\,\sin\,\tau_2; \tag{6}$$

This is essentially the same exponent as $F_{+,-}^{(1)}$ occurring in the calculations of $C_{A1}^{(2)}$ for $\vartheta \neq 0$ or $\pi$ (see $A$. p. 27 ff.), and of course the calculations given there can be reused. We have a saddle point if

$$\tau_1 - \tau_2 = \pm\,\frac{\Theta}{2}\cos\,\psi - \pi N;$$

For $N = 0$ we get an acceptable value for $\tau_1$, close to $\frac{\pi}{2}$, whichever sign we choose before $\Theta$. With the plus sign (the term (0; +)) we get a value just above $\frac{\pi}{2}$; in the term (0; —) we get a value below $\frac{\pi}{2}$. Thus we must take into account a saddle point situated just outside the interval of integration.

From ($A$ 77) and ($A$ 78) applied to the actual case, we get for the saddle point $\overset{+}{\tau}_{10}$ in the term (0; +) and for $\overset{-}{\tau}_{10}$ in the term (0; —):

$$-\cos\overset{+}{\tau}_{10} = \cos\overset{-}{\tau}_{10} = \frac{n\sin\dfrac{\Theta\cos\psi}{2}}{\sqrt{n^2-2n\cos\dfrac{\Theta\cos\psi}{2}+1}}. \tag{7}$$

Remembering that $\Theta$ is a small quantity we can write this:

$$\overset{+;-}{\tau}_{10} = \frac{\pi}{2} \pm \frac{n\,\Theta\cos\psi}{p}. \tag{7'}$$

The denominator $p$ means:

$$p = \sqrt{n^2-2n\cos\frac{\Theta\cos\psi}{2}+1}. \tag{8}$$

In (7′) we have excluded the case $n \sim 1$. We now expand the slowly varying part of the integrand and the exponents $G^{(1)}_{+,-}$ in series from the saddle point. With the temporary notations:

$$\left.\begin{aligned} y_1 &= \tau_1 - \overset{+}{\tau}_{10}; \\ y_2 &= \tau_1 - \overset{-}{\tau}_{10}; \end{aligned}\right\} \tag{9}$$

the result is written

$$\begin{aligned} \frac{\sin^2\tau_1\sin\tau_2\cos\tau_1}{(n\sin\tau_1+\sin\tau_2)^2} &= \frac{1}{(n+1)^2}\left[-\frac{n\,\Theta\cos\psi}{2\,p} - y_1 + \cdots\right] = \\ &= \frac{1}{(n+1)^2}\left[\frac{n\,\Theta\cos\psi}{2\,p} - y_2 + \cdots\right]; \end{aligned} \tag{10}$$

and

$$\left.\begin{aligned} G_s^{(1)}(\tau^s_{10}) &= -2\,(n-1) - \frac{n\,(n-1)}{4\,p^2}\,\Theta^2\cos^2\psi + \cdots; \\ G_s^{(1)\prime\prime}(\tau^s_{10}) &= \frac{2\,(n-1)}{n} + \frac{(n-1)\,(1+n-n^2)}{4\,n\,p^2}\,\Theta^2\cos^2\psi + \cdots; \end{aligned}\right\} \tag{10}$$

for both $s = +$ and $s = -$.

We now perform the inner integrations in (5), omitting all terms but (0; +) and (0; —). The appropriate path of integration consists, for each term, of a straight line of the type ($A$ 71):

$$Re\,(\tau-\tau_{10}) = Im\,(\tau-\tau_{10}); \tag{11}$$

from $\tau - \tau_{10} = -(1+i)\,\infty$ down to the saddle point, and of the part of the real axis between the saddle point and $\tau_1 = \frac{\pi}{2}\cdot$ The contributions from the path (11) are obtained from the formula:

$$\int\limits_{-(1+i)\infty}^{0} y^m e^{i\varrho c y^2} d y = \frac{(-1)^m}{2} \cdot \left(\frac{e^{i\frac{\pi}{2}}}{\varrho c}\right)^{\frac{m+1}{2}} \cdot \Gamma\left(\frac{m+1}{2}\right); \quad (c > 0). \tag{12}$$

Denoting by $R$ the contributions from the remaining piece of the integration path we get:

$$C_{A1}^{(2)} + C_{A1}^{(1)} = \frac{8 n^2 i \varrho e^{-2 i \varrho (n-1)}}{\pi (n+1)^2} \cdot \left(1 + \frac{\Theta^2}{3}\right) \cdot$$

$$\cdot \int\limits_{0}^{\frac{\pi}{2}} \frac{\exp\left(- i \varrho \frac{n(n-1)}{4 p^2} \Theta^2 \cos^2 \psi\right)}{2(n-1) - \frac{(n-1)(n^2-n-1)}{4 p^2} \Theta^2 \cos^2 \psi} d \psi + R. \tag{13}$$

In the denominator of the integrand the second term can be neglected. The error arising from this exclusion is less than 2 % for $\varrho > 45$ and $\Theta < \frac{8}{\varrho}\cdot$ In the following we will use the notation:

$$\beta = \frac{\varrho n (n-1) \Theta^2}{8 p^2}; \tag{14}$$

and thus the expression for $C_{A1}^{(2)} + C_{A1}^{(1)}$ is abridged to:

$$C_{A1}^{(2)} + C_{A1}^{(1)} = \frac{4}{\pi} \cdot i \varrho \cdot e^{-2 i \varrho (n-1)} \cdot \frac{n^2}{(n-1)(n+1)^2} \cdot \left(1 + \frac{\Theta^2}{3}\right) \int\limits_{0}^{\frac{\pi}{2}} e^{-2 i \beta \cos^2 \psi} d \psi + R. \tag{13'}$$

We note that $\beta$ is no large quantity. In particular, for $\Theta < \frac{8}{\varrho}$; $n > 4/3$; $\varrho > 20$, we get:

$$\beta < 1.6; \tag{14'}$$

For $R$ we have:

$$R = \frac{4 n \varrho^2}{\pi} \cdot \frac{e^{-2 i \varrho (n-1)}}{(n+1)^2} \cdot \left(1 + \frac{\Theta^2}{3}\right) \int\limits_{0}^{\frac{\pi}{2}} e^{-2 i \beta \cos^2 \psi} d \psi \cdot$$

$$\cdot \left[2 b \cos \psi \int\limits_{0}^{b \cos \psi} e^{i \varrho a_1 y^2} d y - 2 \int\limits_{0}^{b \cos \psi} y e^{i \varrho a_1 y^2} d y\right] \cdot$$

The new notations mean:

$$a_1 = \tfrac{1}{2} G^{(1)''}(\tau_{10})$$

$$b = \frac{n \Theta}{2 p} \cdot$$

We have used the fact that the terms (0; +) and (0; —) give the same contribution to $R$.

The second of the inner integrals can be calculated at once; this part of $R$ becomes:

$$-\frac{4}{\pi} i \varrho e^{-2i\varrho(n-1)} \cdot \frac{n^2}{(n-1)(n+1)^2}\left(1+\frac{\Theta^2}{3}\right)\int\limits_0^{\frac{\pi}{2}} \left(e^{-2i\beta\cos^2\psi}-1\right) d\psi.$$

Denoting the rest of $R$ by $R'$ we can now write for (13′):

$$C_{A1}^{(2)} + C_{A1}^{(1)} = 2 i \varrho e^{-2i\varrho(n-1)} \cdot \frac{n^2}{(n-1)(n+1)^2} \cdot \left(1+\frac{\Theta^2}{3}\right) + R'; \qquad (15)$$

where:

$$R' = \frac{4n^2\varrho^2\Theta}{\pi(n+1)^2} \cdot e^{-2i\varrho(n-1)} \cdot \left(1+\frac{\Theta^2}{3}\right)\int\limits_0^{\frac{\pi}{2}} d\psi \cdot \frac{\cos\psi}{p} e^{-2i\beta\cos^2\psi} \cdot \int\limits_0^{\frac{n\Theta\cos\psi}{2p}} e^{i\varrho a_1 y^2} dy. \qquad (16)$$

The inner integral is essentially a Fresnel integral:

$$\int\limits_0^{\frac{n\Theta\cos\psi}{2p}} e^{i\varrho a_1 y^2} dy = \frac{1}{\sqrt{\varrho a_1}} \int\limits_0^{\sqrt{2\beta}\cdot\cos\psi} e^{iu^2} du;$$

on which we use the formula due to LOMMEL[1]:

$$\int\limits_0^{\sqrt{2z}} e^{iu^2} du = \sqrt{2z}\, e^{iz} S(z); \qquad (17)$$

where:

$$S(z) = \sum_{m=0}^{\infty} (-1)^m \sqrt{\frac{\pi}{2z}} J_{2m+\frac{1}{2}}(z) - i \sum_{m=0}^{\infty} (-1)^m \sqrt{\frac{\pi}{2z}} J_{2m+\frac{3}{2}}(z); \qquad (17')$$

is a function whose values are readily obtained from the existing rather extensive tables of spherical Bessel functions.[2] For values of $z$ considered in this paper the series in (17′) converge rapidly. The introduction of $S(z)$ takes $R'$ into:

$$R' = \frac{2\varrho}{\pi}\left(1+\frac{\Theta^2}{3}\right) \cdot \frac{n^3}{(n+1)^2} e^{-2i\varrho(n-1)} \cdot \varrho\,\Theta^2 \int\limits_0^{\frac{\pi}{2}} \frac{\cos^2\psi}{p^2} e^{-i\beta\cos^2\psi} S(\beta\cos^2\psi)\, d\psi. \qquad (18)$$

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[1] See e. g. WATSON, Theory of Bessel Functions, p. 546.

[2] Tables of Spherical Bessel Functions, I & II, M. T. P., 1947.

We can at once see that for very small values of $\Theta$ this integral tends to $\frac{\pi}{4(n-1)^2}$.

To get a description for a wider range of $\Theta$ we proceed as follows. Numerical checking confirms that for $\varrho\,\Theta < 8$ and $\varrho > 20$, we can use the approximation:

$$S(z) \approx \sqrt{\frac{\pi}{2z}}\,[J_{\frac{1}{2}}(z) - i\,J_{\frac{3}{2}}(z) - J_{\frac{5}{2}}(z)]. \tag{19}$$

For $\varrho\,\Theta < 4$ we may even drop the last term. We denote the contribution to the integral in (18) from the first two terms by $I_1$ and from the third by $I_2$. We get:

$$\left.\begin{aligned} I_1 &= \frac{i}{p^2\beta}\int_0^{\frac{\pi}{2}}\left[e^{-2i\beta\cos^2\psi} - e^{-i\beta\cos^2\psi}\,.\,\frac{\sin(\beta\cos^2\psi)}{\beta\cos^2\psi}\right]d\psi; \\ I_2 &= \frac{1}{p^2\beta}\int_0^{\frac{\pi}{2}} e^{-i\beta\cos^2\psi}\left[\sin(\beta\cos^2\psi) + \frac{3\cos(\beta\cos^2\psi)}{\beta\cos^2\psi} - \frac{3\sin(\beta\cos^2\psi)}{\beta^2\cos^4\psi}\right]d\psi \end{aligned}\right\} \tag{20}$$

as $p^2\beta$ is independent of $\psi$. In order to be able to compute these integrals without resorting to numerical integration we now suppose $\beta$ to be independent of $\psi$. This must be considered tolerable as the dependance on $\psi$ is weak, $\beta$ lying between the limits:

$$\frac{\varrho\,n\,(n-1)\,\Theta^2}{8\left(n^2 - 2n\cos\frac{\Theta}{2} + 1\right)} < \beta < \frac{\varrho\,n\,\Theta^2}{8\,(n-1)}.$$

We will use the value for $\psi = 0$:

$$\beta \approx \frac{\varrho\,n\,(n-1)\,\Theta^2}{8\left(n^2 - 2n\cos\frac{\Theta}{2} + 1\right)}. \tag{21}$$

It is now convenient to use the expansions:

$$\left.\begin{aligned} \frac{\sin z}{z} &= \sum_{m=0}^{\infty} a_m \cos\left(m + \tfrac{1}{2}\right) z; \\ \frac{\cos z}{z} - \frac{\sin z}{z^2} &= \sum_{m=0}^{\infty} b_m \sin\left(m + \tfrac{1}{2}\right) z; \end{aligned}\right\} \tag{22}$$

where:

$$-\frac{b_m}{m+\frac{1}{2}} = a_m = \frac{Si\,(m+\frac{3}{2})\,\pi - Si\,(m-\frac{1}{2})\,\pi}{\pi}. \tag{22'}$$

They transform $I_1$ and $I_2$ into series of integrals of the type:

$$I_x = \int\limits_0^{\frac{\pi}{2}} e^{2ix\cos^2\psi}\, d\psi; \tag{23}$$

which is easily brought back to integral representations for the Bessel function of the order zero. We get:

$$I_x = \frac{\pi}{2} e^{ix} J_0(x). \tag{24}$$

Applying this result, introducing the new expressions for $I_1$ and $I_2$ in $R'$, and for $R'$ in (15) we obtain:

$$C^{(2)}_{A1} + C^{(1)}_{A1} = 2i\varrho\, e^{-2i\varrho(n-1)} \cdot \left(1 + \frac{\Theta^2}{3}\right) \frac{n^2}{(n-1)(n+1)^2}\chi; \tag{25}$$

where:

$$\left.\begin{aligned}
\chi &= 1 + 4\chi_1 + 2\chi_2;\\
\chi_1 &= e^{-i\beta} J_0(\beta) - \tfrac{1}{2}\sum_{m=0}^{\infty} a_m \left[ e^{i\frac{2m-1}{4}\beta} J_0\left(\frac{2m-1}{4}\beta\right) + \right.\\
&\qquad\qquad \left. + e^{-i\frac{2m+3}{4}\beta} J_0\left(\frac{2m+3}{4}\beta\right)\right];\\
\chi_2 &= e^{-i\beta} J_0(\beta) - 1 - 3\sum_{m=0}^{\infty} b_m \left[ e^{i\frac{2m-1}{4}\beta} J_0\left(\frac{2m-1}{4}\beta\right) - \right.\\
&\qquad\qquad \left. - e^{-i\frac{2m+3}{4}\beta} J_0\left(\frac{2m+3}{4}\beta\right)\right];
\end{aligned}\right\} \tag{25'}$$

Performing the same calculations for the other $C_1$ terms we see that for $\Theta \sim 0$ and with the accuracy of (25):

$$C^{(\nu)}_{A1} = - C^{(\nu)}_{B1}; \qquad \nu = 1;\ 2. \tag{26}$$

Therefore, (25) at once yields the expressions required to give the field components:

$$\left.\begin{aligned}
E''_{\vartheta 1} &= \frac{2a}{r}\cos\varphi\, e^{-ik_0[r+2a(n-1)]} \cdot \frac{n^2}{(n-1)(n+1)^2}\left(1 + \frac{\Theta^2}{3}\right)\chi;\\
E''_{\varphi 1} &= E''_{\vartheta 1} \cdot \operatorname{tg}\varphi.
\end{aligned}\right\} \tag{27}$$

For very small angles $\Theta$ we get the considerable simplification:

$$E''_{\vartheta 1} = \frac{2a}{r}\cos\varphi\, e^{-ik_0[r+2a(n-1)]} \cdot \frac{n^2}{(n-1)(n+1)^2}\left(1 - i \cdot \frac{n}{4(n-1)}\varrho\, \Theta^2\right) \cdot \tag{27'}$$

We can check our expressions (27) by comparing them to the formula given by MECKE.[1] For $\Theta = 0$ we get an intensity:

$$\eta_0 = \frac{4\,n^4}{(n-1)^2\,(n+1)^4} \cdot \left(\frac{a}{r}\right)^2; \tag{28}$$

whereas Mecke gives[2]:

$$\eta_{0,\ \text{Mecke}} = \frac{4\,n^2}{(n-1)^2\,(n+1)^4} \cdot \left(\frac{a}{r}\right)^2. \tag{29}$$

The comparison between the two expressions must favour (28). The original expressions for the $C_1$ terms (cf. eq. ($A$ 25)) show that the field components tend to zero as $\frac{1}{n}$ for $n \to \infty$; this accords with (28) but not with (29). The error in Mecke's deduction is, in fact, easily detected. Setting up eq. (17) p. 631, he uses the Fresnel formulae as though the two refractions were identical. They are not. Changing Mecke's expressions accordingly[3] we get, in eq. (18) p. 631, a factor $\frac{4\,n}{(n+1)^2}$ instead of $\frac{4}{(n+1)^2}$; the additional $n$ in the numerator then transforms (29) to (28).

For $\Theta \neq 0$ other differences between our expressions and Mecke's seem to occur. From his eq. (21) on p. 632 there follows, for small values of $\Theta$, an intensity dependance on $n$ and $\Theta$ only, whereas our formula seems to give a dependance on $\varrho$ as well. A closer investigation shows, however, that there is no difference of this kind in the two representations. A numerical computation of $\chi$ tells us that the intensity is almost constant for small values of $\Theta$; so does Mecke's formula. In fact, for small values of $\Theta$, it is easily proved that (27′) might be given as a result of Mecke's expressions as well (if, of course, the error pointed out is corrected).

## 3. The reflected radiation

The additional complication already mentioned and arising in the treatment of the terms $C_{02}$ must be considered fatal to the method used until now. It cannot but meet a failure when the asymptotic expressions are unreliable in the vicinity of the presumptive saddle point. Our defense in $A$, pp. 8—9, of the application of such expressions in a »wrong region» does not essentially change this situation. What we said there can be upheld only if the change of the phase for $l \sim \varrho$ is rapid enough to cancel the major part of these terms; this must be the case both in the exact series and in our in this case quasi-asymptotic substitute for it.

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[1] MECKE, Ann. d. Ph., *62*, pp. 631 & 632, 1920.

[2] We have corrected an obvious inadvertancy of Mecke in passing from eq. (21) to (22).

[3] Eq. (17) will then run (Mecke's notations):

$$a_s = a \cdot \frac{4\,n\left(n\cos\frac{\varphi}{2} - 1\right)\left(n - \cos\frac{\varphi}{2}\right)}{(n^2-1)^2};$$

Realising this breakdown of the method the author has nothing better to suggest than a direct summation. As the available tables of the functions involved do not allow us, for large values of $\varrho$[1], to begin with the rigorous expressions, we must introduce asymptotic formulae in the same way as in *A*. We start with the form (*A* 18) for the $C_{02}$ terms and apply (*A* 20):

$$\left.\begin{aligned} C^{(2)}_{X\,02} &= \sum_{l=1}^{\varrho-1} \frac{l+\frac{1}{2}}{l(l+1)} \cdot F_{X\,02} \cdot P_l^{(1)\prime}(\cos\Theta)\sin\Theta; \\ X &= A \text{ or } B. \end{aligned}\right\} \tag{30}$$

The terms with an upper index (1) are obtained through the same exchange as in (2). The new notations mean:

$$\left.\begin{aligned} F_{A\,02} &= \frac{H^{(1)}_{l+\frac{1}{2}}(\varrho)}{H^{(2)}_{l+\frac{1}{2}}(\varrho)} - \frac{2\,i\,n}{[\zeta_l^{(2)}(\varrho)]^2\,[q_{1\,l}(n\,\varrho) - n\,q_{2\,l}(\varrho)]}; \\ F_{B\,02} &= -\frac{H^{(1)}_{l+\frac{1}{2}}(\varrho)}{H^{(2)}_{l+\frac{1}{2}}(\varrho)} + \frac{2\,i}{[\zeta_l^{(2)}(\varrho)]^2\,[n\,q_{1\,l}(n\,\varrho) - q_{2\,l}(\varrho)]}. \end{aligned}\right\} \tag{31}$$

We now introduce the Debye approximations for the Bessel functions and the approximations (*A* 99) for the Legendre functions. We then get expressions of the type:

$$\left.\begin{aligned} C^{(1)}_{X\,02} &= -\varrho\frac{\Theta}{\sin\Theta}\left(1+\frac{\Theta^2}{8}\right)[S^{(1)}_{X\,02} + R^{(1)}_{X\,02}]. \\ C^{(2)}_{X\,02} &= \varrho\left[S^{(1)}_{X\,02} + R^{(1)}_{X\,02} - \left(1+\frac{\Theta^2}{12}\right)(S^{(2)}_{X\,02} + R^{(2)}_{X\,02})\right]. \end{aligned}\right\} \tag{32}$$

Here the $S'$s mean the contributions from terms with $l < \varrho - [\sqrt[3]{\varrho}]$ and the $R'$s from the terms in the interval $\varrho - [\sqrt[3]{\varrho}] \leq l \leq \varrho - 1$.

The numerical computation of $S^{(\nu)}_{X\,02}$; $\nu = 1$ or $2$; meets no theoretical difficulties but is, of course, a very tedious piece of work. As a consequence of the absence of saddle points in the interval $(1;\ \varrho - [\sqrt[3]{\varrho}])$ for $l$ we must expect the sums $S_{X\,02}$ to be of the same magnitude as a single term in the series. The calculations confirm this supposition; they have been carried out for $\varrho = 20$; $40$; $60$; $n = 4/3$; and a dozen values of $y = \varrho\,\Theta$.

This is quite in accordance with the result obtained with the saddle point method. A fairly simple calculation shows that the asymptotic value thus afforded is zero of the contribution from the decisive point $l = 0$. The result also agrees with the geometrical optical picture given in *A*. A term with a certain value of $l$ corresponds to rays with an angle of incidence $\alpha$ defined by:

$$\sin\alpha = \frac{l+\frac{1}{2}}{\varrho}. \tag{33}$$

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[1] An upper limit for an exact calculation lies, for $n=4/3$, at about $\rho=18$. Above this value the numerical work increases enormously, as we must then compute the Bessel functions ourselves.

We have therefore obtained the quite natural information that, for low values of $\alpha$, the reflected part of the ray does not contribute to the intensity in the forward direction.

There remains to calculate the influence from the few terms with $l \geq \varrho - [\sqrt[3]{\varrho}]$. Introducing the approximations ($A$ 99) for the Legendre functions, we get:

$$\left.\begin{aligned} R^{(1)}_{X02} &= \sum_{\eta=1}^{q} F_{X02} \cdot \frac{J_1[y - (\eta - \frac{1}{2})\,\Theta]}{y}; \\ R^{(2)}_{X02} &= \sum_{\eta=1}^{q} F_{X02} \left(1 - \frac{\eta - \frac{1}{2}}{\varrho}\right) J_0[y - (\eta - \frac{1}{2})\,\Theta]; \\ \eta &= \varrho - l; \quad q = [\sqrt[3]{\varrho}]. \end{aligned}\right\} \tag{34}$$

It would be a great advantage to write these sums with the dependance on $\Theta$ outside the summation sign. Gegenbauer's addition formula for Bessel functions[1] offers a possibility for this; for $J_0$ and $J_1$ it runs:

$$\left.\begin{aligned} J_0(t) &= J_0(u)\,J_0(v) + 2\sum_{m=0}^{\infty} J_m(u)\,J_m(v)\cos m\alpha; \\ \frac{J_1(t)}{t} &= 2\sum_{m=1}^{\infty} \frac{J_m(u)}{u} \cdot \frac{J_m(v)}{v} \cdot \frac{d\cos m\alpha}{d\cos\alpha}; \end{aligned}\right\} \tag{35}$$

where:

$$t^2 = u^2 - 2uv\cos\alpha + v^2. \tag{36}$$

We observe that we can write the argument of the Bessel functions approximately:

$$t = \Theta(\varrho - \eta + \tfrac{1}{2}) \approx y\sqrt{1 - \frac{2(\eta - \frac{1}{2})}{\varrho} + \frac{(q + \frac{1}{2})^2}{\varrho^2}}. \tag{37}$$

Introducing the new notations:

$$\left.\begin{aligned} x &= \frac{q + \frac{1}{2}}{\varrho}\,y; \\ s_{Xm} &= \sum_{\eta=1}^{q} \left(\frac{\eta - \frac{1}{2}}{q + \frac{1}{2}}\right)^m \cdot \left(1 - \frac{\eta - \frac{1}{2}}{\varrho}\right) F_{X02}; \end{aligned}\right\} \tag{38}$$

we then get

$$\left.\begin{aligned} R^{(1)}_{X02} &= 2\,\frac{J_1(x)\,J_1(y)}{xy}\,s_{X0} + 8\,\frac{J_2(x)\,J_2(y)}{xy}\,s_{X1}; \\ R^{(2)}_{X02} &= [J_0(x)\,J_0(y) - 2\,J_2(x)\,J_2(y)]\,s_{X0} + 2\,J_1(x)\,J_1(y)\,s_{X1} + \\ &\qquad + 4\,J_2(x)\,J_2(y)\,s_{X2}; \end{aligned}\right\} \tag{39}$$

neglecting terms with a factor $J_m(x)$ for $m \geq 3$.

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[1] See WATSON, Theory of Bessel Functions, p. 362.

We must now find appropriate expressions for $F_{X02}$ as functions of $\eta$, $n$ and $\varrho$. Leaving aside the case $n \approx 1$, we can still use ($A$ 27) for $q_{1l}(n\varrho)$, giving

$$q_{1l}(n\varrho) \approx i\frac{\sqrt{n^2-1}}{n}\left[1+\frac{\eta-\frac{1}{2}}{\varrho(n^2-1)}\right]. \tag{40}$$

We further use the recurrence formula:

$$\zeta_l'(\varrho) = \frac{l+1}{\varrho}\zeta_l(\varrho) - \zeta_{l+1}(\varrho); \tag{41}$$

and thus get, omitting the argument $\varrho$ of the Bessel functions:

$$\left.\begin{aligned} F_{A02} &= \frac{H^{(1)}_{l+\frac{1}{2}}}{H^{(2)}_{l+\frac{1}{2}}} - \frac{4in}{\pi\varrho\,[H^{(2)}_{l+\frac{1}{2}}]^2\cdot T_1}; \\ F_{B02} &= -\frac{H^{(1)}_{l+\frac{1}{2}}}{H^{(2)}_{l+\frac{1}{2}}} + \frac{4i}{\pi\varrho\,[H^{(2)}_{l+\frac{1}{2}}]^2\cdot T_2}. \end{aligned}\right\} \tag{42}$$

Here $T_1$ and $T_2$ denote:

$$\left.\begin{aligned} T_1 &= n\frac{H^{(2)}_{l+\frac{3}{2}}}{H^{(2)}_{l+\frac{1}{2}}} - n + \frac{i}{n}\sqrt{n^2-1} + \frac{i}{n\varrho}\cdot\frac{\eta-\frac{1}{2}}{\sqrt{n^2-1}} + \frac{n(\eta-1)}{\varrho}; \\ T_2 &= \frac{H^{(2)}_{l+\frac{3}{2}}}{H^{(2)}_{l+\frac{1}{2}}} - 1 + i\sqrt{n^2-1} + \frac{i}{\varrho}\frac{\eta-\frac{1}{2}}{\sqrt{n^2-1}} + \frac{\eta-1}{\varrho}. \end{aligned}\right\} \tag{43}$$

To compute the numerical values of the Hankel functions we use the Debye formula for the transitional region[1] to get $H^{(\nu)}_{\varrho\pm\frac{1}{2}}(\varrho)$:

$$\left.\begin{aligned} H^{(1)}_{\varrho\pm\frac{1}{2}}(\varrho) &= 0.246163\,z \mp 0.062214\,z^2 \mp 0.001026\,z^4 + 0.000352\,z^5 - \\ &\qquad - 0.000044\,z^7 + \cdots + \\ &+ i\,[-0.426366\,z \mp 0.107758\,z^2 \pm 0.001777\,z^4 + 0.000609\,z^5 + \\ &\qquad + 0.000076\,z^7 + \cdots]; \\ z &= \sqrt[3]{\frac{6}{\varrho}}; \end{aligned}\right\} \tag{44}$$

and the recurrence formulae for cylinder functions to get the rest.

Accordingly we calculate the quantities $R$ via the $s_{X_m}'$s. The result is now, that in the forward direction the reflected radiation (provided the interpretation given in $A$ is maintained) falls far short of the refracted. As an example we take $y = 3.8$; $\varrho = 60$. The moduli of both $C^{(2)}_{A02}$ and $C^{(2)}_{B02}$ are about $0.27\,\varrho$. This is less than $\frac{1}{7}$ of the refracted radiation in the same direction.

This result does not agree with Mecke's. His result is considerably larger. The difference is not surprising. The impossibility of using the saddle point method here must indicate that geometrical optics can describe the phenomenon but inadequately.

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[1] See WATSON, Theory of Bessel Functions, p. 245.

There is another point worth noticing in the numerical results for $C_{02}$. The real part of $C_{02}$ is much less than the imaginary; this corresponds to a dominance of the real part of the field components and agrees with geometrical optics, i. e. the expressions in Mecke's paper yield the same phases.

A grave objection can be raised to the calculations performed in this paragraph. It bears upon the carrying out of the summation only to $l = \varrho - 1$. The next higher terms must have some influence on the phenomenon, in any case somewhat larger than the contribution from the reflected radiation. The author has persisted in performing the calculation to this fixed limit wholly from physical reasons. For the terms above this limit we lack a correspondence in the geometrical optical picture; to calculate the influence from them must therefore be performed from the original expressions, before the splitting up leading to the physical interpretation. Apart from this, the calculation can be carried out in the manner shown in this paragraph for the quantities $R$, but it is tedious. We will come back to this question in the next paragraph.

## 4. On the positions of the minima

We can actually use the expressions ($A$ 102) for the diffracted radiation unaltered. The only point to discuss is the passing from ($A$ 100) to ($A$ 102), but the exchange, if performed, is unimportant. We will in this paragraph be interested in the positions of the minima of intensity. Numerical checking shows that even for $\varrho = 25$ the approximation ($A$ 102) is safe.

From ($A$ 102) and (27) we can see that the sphere has no polarizing effect on the refracted and diffracted radiation in the forward direction. Thus we need consider one component of $E$ only. To get the simplest possible description we will furthermore neglect the influence from the reflected radiation; we can do this for two reasons, firstly the smallness of the terms $C_{02}$, secondly the fact that the reflected radiation has a phase differing about $\frac{\pi}{2}$ from that of the diffracted radiation.

Then we can write:

$$E''_{\vartheta 01} + E''_{\vartheta 1} \doteq i \frac{a}{r} \cos \varphi \cdot e^{-i k_0 r} (F_{01} + F_1); \tag{45}$$

where:

$$\left.\begin{aligned} F_{01} &= \varrho \frac{J_1(y)}{y}; \\ F_1 &= -2 i e^{-2 i \varrho (n-1)} \cdot \frac{n^2}{(n-1)(n+1)^2} \chi . \end{aligned}\right\} \tag{46}$$

So we have to seek minima for:

$$I = | F_{01} + F_1 |^2 \tag{47}$$

The first step to do this, the tabulation of $\chi$ as a function of $\beta$, reveals the important relation:

$$|\chi| \approx 1 \tag{48}$$

already utilized in the preceding paragraph. The difference for $\beta < 1.5$ never exceeds 2 %, and the influence from the performed approximations must be considered larger. The physical meaning of (48) is that in the forward direction, up to the angle giving the second minimum, the intensity of the refracted radiation is practically constant, but that the phase changes.

On the other hand, the phase is a much more rapidly varying function of $\varrho$, this being a consequence of the factor $e^{-2i\varrho(n-1)}$. In comparison with the rapid fluctuation mentioned we can, for large particles (corresponding to low values of $\beta$), consider $\chi$ independent of $\Theta$, using its value for $\beta = 0$:

$$\chi_{\beta=0} = 1\,. \tag{49}$$

Then we get the following simple picture. The minima are situated somewhere around the values given by (1 a) in the intervals defined by:

$$\left|\frac{J_1(y)}{y}\right| < \frac{1}{\varrho}\cdot\frac{2\,n^2}{(n-1)\,(n+1)^2}\,. \tag{50}$$

For increasing $\varrho$ or $n$ these intervals decrease; we also see that the observed minima are situated on one side of $y_m$ (as given by (1 a)) when $m$ is odd, and on the other when $m$ is even. Thus, in determining the radius of a large particle from the first two minima, using the »ideal» eq. (1 a), we get one value that is too large and another too small.

It may interest to see the effect of an application of this rather rough approximation on an experimental material. We choose the measurements on mists and clouds given in the extensive treatise of KÖHLER already cited. Starting with the value $\varrho_1 = \dfrac{2\pi r_1}{\lambda}$ obtained from the first minimum with the ideal eq. (1 a) ($r_1$ = the value of the radius given in Köhler's treatise) we try to find a better one, $\varrho_1'$, being the least zero of the equation:

$$\frac{J_1\left(\dfrac{\varrho_1'}{\varrho_1}\cdot 3.832\right)}{\dfrac{\varrho_1'}{\varrho_1}\cdot 3.832} - \frac{1}{\varrho_1'}\cdot\frac{2\,n^2}{(n-1)\,(n+1)^2}\cdot \sin 2\,\varrho_1'\,(n-1) = 0. \tag{51}$$

Afterwards we check our result by computing the effect of the corresponding correction on $\varrho_2$, obtained from the second minimum. From the table 4 in the cited treatise we get for the series 15–26[1], that in 8 cases out of 9 the correction goes in the right direction; in three other cases the correction is small. An investigation of the series 69—90[1] in table 7 (measurements below 0° C) gives the same result. Leaving aside 6 series in which the mean error exceeds 2 %, we get a positive result in 11 cases and a negative in 2; in three

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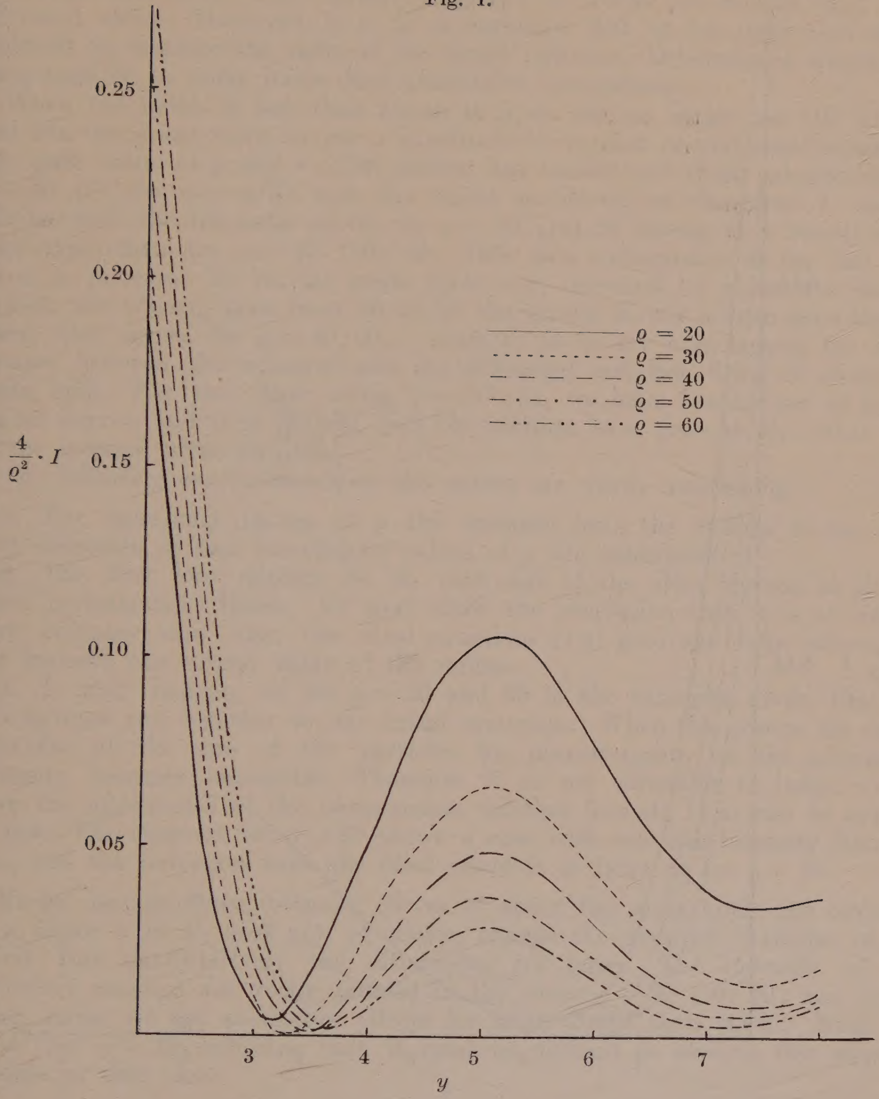
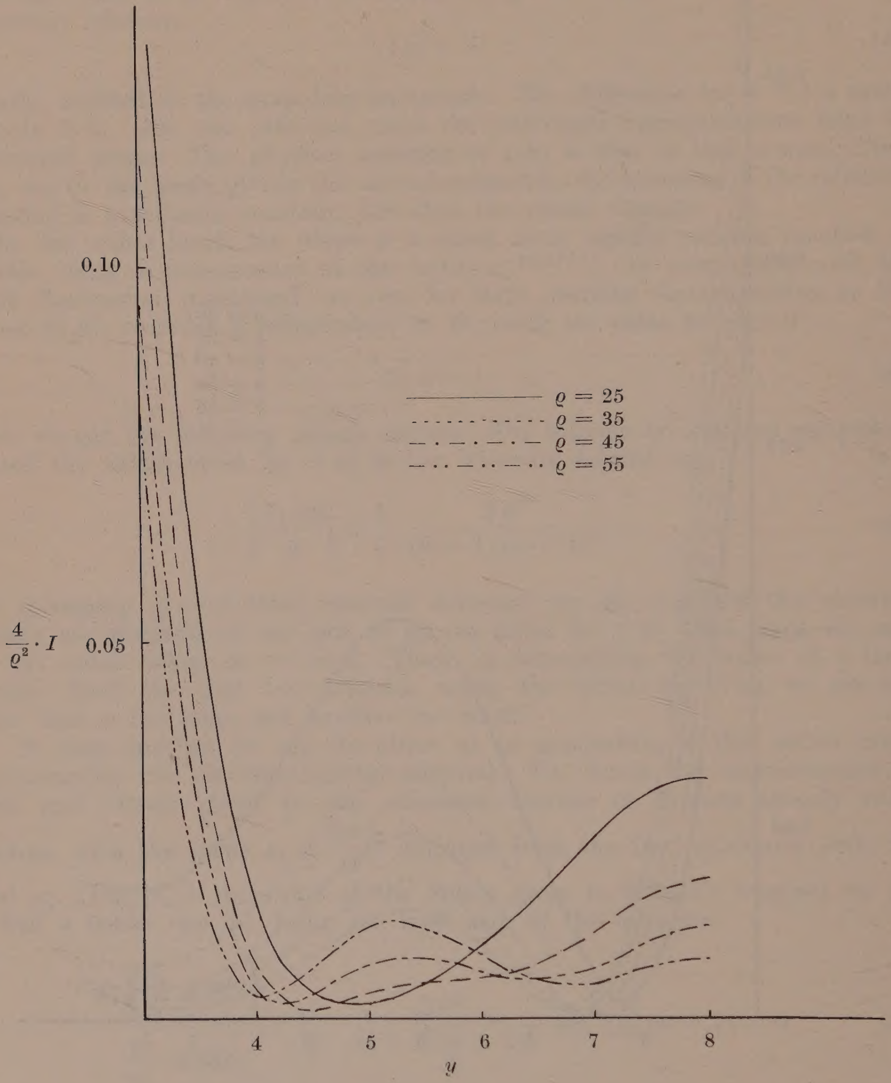
[1] We only consider series with $r > 11\,\mu$.

Fig. 1.



cases the change is small. This agreement is surprisingly good; in an investigation on natural mists and clouds we must expect significant errors from unforeseen sources, e. g. lacking homogeneity in the mist, or changes of its state during the observations. Further, our theory has been based on the assumption of a monochromatic radiation. In his observations Köhler uses white light, in the calculations applying a mean of the wave lengths. Maintaining that the agreement is better than could have been expected, we must at the same time remember, that the comparison does not conclusively confirm our theory. The uncertainty of the used material is too large.

Fig. 2.



We can find another conclusion drawn by Köhler and supporting our simple picture. If we investigate a great many spheres, varying in size in an irregular manner and if we compute the values $r_1$ and $r_2$ from the first two minima by the ideal formula, then our description requires the mean of $\frac{r_1 - r_2}{r_2}$ to be very small. This is exactly what Köhler gets and he interprets it as an effect of random errors only.

We can also compare with the experimental work of MECKE, loc. cit. On pp. 490 & 491 he gives a qualitative description wholly agreeing with what has been said here. The spheres measured by Mecke are smaller than those discussed above. Moreover, it is to be regretted that he has only used optical methods to measure the radii of the liquid particles. Otherwise it would have been possible to make more than qualitative comparisons.

When the radius is less than about 10 $\mu$ we can no longer use (49) in (47) and the numerical work to get a substitute for it must be performed separately for each value of $\varrho$ and $n$. The author has carried out these calculations for $\varrho = 20$ (5) 60; $n = 4/3$; and the result is shown in diagrams 1 and 2. We see that the intensity curves for $\varrho = 20$ (10) 60 belong to a totally different type than for $\varrho = 25$ (10) 55. This is a consequence of the fact that when $\varrho$ increases by 10, the angle $2\varrho(n-1)$ increases by about $2\pi$ when $n$ is $4/3$, but when $\varrho$ goes from 20 to 25 the growth is only a little more than $\pi$. Then, the curves for $\varrho = 20, 30, \ldots$ are so as to say homologous, the interference between the refracted and the diffracted radiation being of about the same kind. For the other series, $\varrho = 25$ etc., we have another set of homologous curves, and it is natural that the minima have gone to the other part of the interval given by (50).

The following conclusions from the curves are worth mentioning.

1. For increasing values of $\varrho$ the distance from the minima to the ideal ones decreases, if two homologous values of $\varrho$ are considered.
2. The first two minima lie on each side of the ideal minima in all the cases investigated. Hence, we may draw the conclusion that it is at least a very ordinary case, that the ideal equations (1 a) give one value above and one beneath the correct value of the radius.
3. It may happen, as for $\varrho = 25$ and 35 in the examples given, that the two minima run together to one broad minimum. When this occurs, the determination of the size of the particles by measurements on the minima of intensity becomes impossible. Therefore it is not advisable to judge merely from the appearance of the phenomenon whether formula (1 a) can be applied or not. The diagram for $\varrho = 20$ shows a case with »normal» intensity distribution, but the deviation from the ideal curve is as large as for $\varrho = 25$.

Mecke also produces intensity curves of about the same kind: the omission of a factor $n$ in $F_1$ does not, of course, change the essential character of the curve. But quantitatively the differences are large. The intensity in the secondary maxima are about doubled in the series $\varrho = 20$ (10) 60; and in the other series we get anomalous effects for larger drops than Mecke. Even the curve for $\varrho = 45$ indicates that it must be difficult to observe two separate minima in that case.

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We will now sum up what we can say as to the possibility of determining the radius of the sphere. For water ($n = 4/3$) it seems possible to obtain two values, one on each side of the correct one, whenever two separate minima are visible. This is always the case when $\varrho$ exceeds 50. But this determination does not give a value as accurate as would allow the experimental data alone. To reach the best possible value we cannot restrict ourselves to observing the position of the minima only. It must be emphasized that the investigation peformed in this paper has used approximations that are perhaps very severe for $\varrho < 50$. Especially this is the case for increasing $\Theta$, that is in the region of the second minimum.

We will therefore take up the method suggested by Mecke, and using the colours of the phenomenon for $\Theta = 0$ (the aureoles). The determination of these colours must be performed by means of Maxwell's colour triangle in some modified form; the method used by KÖHLER[1] seems to be appropriate. As our expressions for the refracted radiation differs from Mecke's, there must also result differences in the determined colours. Especially, we must expect coloured central maxima for larger drops than foreseen by Mecke. This is in accordance with the results of Köhler; he has obtained coloured aureoles already for $r_1 \sim 15\,\mu$.

In this paper we do not intend to carry out these calculations as they mean a great deal of numerical work. To be on the safe side, we can no more neglect the contribution from the terms $C_{02}$, nor can we avoid a discussion of the terms with $l$ surpassing $\varrho$. All necessary means for carrying out the rather tedious calculations are indicated in the preceding paragraph. It may be left to someone acquainted with the experimental side of the problem to judge, whether the importance of a solution justifies the necessary numerical pains.

I wish to express my gratitude to Professor IVAR WALLER for suggesting this investigation and for his continued interest during the work. I am also indebted to Professor HILDING KÖHLER for helpful discussions of the experimental side of this problem.

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[1] KÖHLER, Ark. Mat. Astr. Fys., *24* A, n:o 9, 1934.

## Errata in A

P. 3, line 6, *delete* zero; p. 4, line —12, *for* Sinclair and Houghton, *read* Stratton and Houghton; p. 11, line 14, *for* sin $\vartheta$ *in the denominator read* $l \sin \vartheta$; p. 20, line 2, *for* $\sqrt{\frac{2}{\varrho}}$, *read* $\sqrt[3]{\frac{2}{\varrho}}$; p. 21, line —9, *for* $(\vartheta - \vartheta_\mu \sqrt{\ }$, *read* $(\vartheta - \vartheta_\mu) \sqrt[3]{\ }$; p. 21, line —7, *for* $\frac{2\pi^2}{3}$, *read* $\frac{2\pi^2}{9}$; p. 23, line —5, *after* $\sum$, *insert* $(-1)^l$; p. 26, line 14, *for* $-\frac{F^{(\mathrm{IV})}(x_0)\, g(x_0)}{[F''(x_0)]^2}$, *read* $-\frac{F^{(\mathrm{IV})}(x_0)\, g(x_0)}{4\,[F''(x_0)]^2} + \frac{5 \cdot [F'''(x_0)]^2 g(x_0)}{12 \cdot [F''(x_0)]^3}$; p. 34, line —10, *for* $\vartheta$, *read* $\Theta$; p. 35, lines 26—27, *for* $\varrho^{\frac{3}{2}}$ *read* $\varrho$; *delete* but zero for $\vartheta \sim \pi$.

Tryckt den 24 februari 1949

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